Olimpiada de Matematică –etapa locală- Galaţi

16 februarie-2020

Clasa a IX-a

Barem de evaluare

- **Pentru orice soluţie corectă, chiar dacă este diferită de cea din barem, se acordă punctajul maxim corespunzător.**
- **Nu se acordă fracţiuni de punct, dar se pot acorda punctaje intermediare pentru rezolvări parţiale, în limitele punctajului indicat în barem.**

Nr. problemei	**Soluţie, rezolvare**	**Puncta**
1.	a) Se aplică inegalitatea mediilor: $\frac{a+b}{2} \geq \sqrt{a \cdot b} \Rightarrow a+b \geq 2 \cdot \sqrt{a \cdot b}$; La fel $b+c \geq 2 \cdot \sqrt{b \cdot c}$, $a+c \geq 2 \cdot \sqrt{a \cdot c}$. Înmulţind aceste relaţii, se obţine $(a+b) \cdot (b+c) \cdot (c+a) \geq 8 \cdot a \cdot b \cdot c$.	**2p**
	b) $a^3+b^3+c^3-3 \cdot a \cdot b \cdot c = (a+b)^3 - 3ab(a+b) + c^3 - 3abc =$ $(a+b)^3 + c^3 - 3ab(a+b+c) =$	**1p**
	$(a+b+c) \cdot \left[(a+b)^2 - c(a+b) + c^2\right] - 3ab(a+b+c) =$ $(a+b+c) \cdot \left(a^2+2ab+b^2+c^2-ac-bc-3ab\right) =$	**1p**
	$(a+b+c) \cdot \left(a^2+b^2+c^2-ac-bc-ab\right) =$ $\frac{1}{2} \cdot (a+b+c) \cdot \left[(a-b)^2+(a-c)^2+(b-c)^2\right] \geq 0, (\forall) a,b,c > 0.$	**1p**
	c) $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} = \frac{a^2}{ab+ac}+\frac{b^2}{bc+ba}+\frac{c^2}{ca+cb} \overset{T.A}{\geq}$ $\frac{(a+b+c)^2}{2 \cdot (ab+ac+cb)} \geq \frac{3 \cdot (ab+ac+cb)}{2 \cdot (ab+ac+cb)} = \frac{3}{2}.$	**2p**

2.	$a)$Fie $[a]=[b]=k,\ k\in\mathbb{Z}\Rightarrow k\le a<k+1$ şi $k\le b<k+1$. $k\le b<k+1\Rightarrow -k-1<-b\le -k$ $\left.\begin{array}{l}-k-1<-b\le -k\\ k\le a<k+1\end{array}\right\}\overset{+}{\Rightarrow}-1<a-b<1\Leftrightarrow \lvert a-b\rvert<1.$	**3p**
	$b)\left[\dfrac{x-1}{2}\right]=\left[\dfrac{x+1}{3}\right]\Rightarrow\left\lvert\dfrac{x-1}{2}-\dfrac{x+1}{3}\right\rvert<1\Leftrightarrow\left\lvert\dfrac{x-5}{6}\right\rvert<1\Leftrightarrow$ $\lvert x-5\rvert<6\Leftrightarrow -6<x-5<6\Rightarrow -1<x<11\Rightarrow 0<x+1<12\Rightarrow$	**1p**
	$\left.\begin{array}{l}0<\dfrac{x+1}{3}<4\\ \left[\dfrac{x+1}{3}\right]=k,k\in\mathbb{Z}\end{array}\right\}\Rightarrow k\in\{0,1,2,3\}.$	**1p**
	1. k=0 $\left.\begin{array}{l}\Rightarrow\dfrac{x+1}{3}\in(0,1)\Rightarrow x\in(-1,2)\\ \dfrac{x-1}{2}\in[0,1)\Rightarrow x\in[1,3)\end{array}\right\}\overset{\cap}{\Rightarrow}x\in[1,2);$ 2. k=1 $\left.\begin{array}{l}\Rightarrow\dfrac{x+1}{3}\in[1,2)\Rightarrow x\in[2,5)\\ \dfrac{x-1}{2}\in[1,2)\Rightarrow x\in[3,5)\end{array}\right\}\overset{\cap}{\Rightarrow}x\in[3,5)$ 3. k=2 $\left.\begin{array}{l}\Rightarrow\dfrac{x+1}{3}\in[2,3)\Rightarrow x\in[5,8)\\ \dfrac{x-1}{2}\in[2,3)\Rightarrow x\in[5,7)\end{array}\right\}\overset{\cap}{\Rightarrow}x\in[5,7)$ 4. k=3 $\left.\begin{array}{l}\Rightarrow\dfrac{x+1}{3}\in[3,4)\Rightarrow x\in[8,11)\\ \dfrac{x-1}{2}\in[3,4)\Rightarrow x\in[7,9)\end{array}\right\}\overset{\cap}{\Rightarrow}x\in[8,9);$ *Soluţia* $: S=[1,2)\cup[3,5)\cup[5,7)\cup[8,9)=[1,2)\cup[3,7)\cup[8,9).$	**2p**

3.	Pentru $n=1 \Rightarrow \sqrt{x_2}+\sqrt{x_1}=\frac{2}{3} \Rightarrow \sqrt{x_2}=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}=\frac{1}{2\cdot 3}$; $n=2 \Rightarrow \sqrt{x_3}+\sqrt{x_2}=\frac{1}{4} \Rightarrow \sqrt{x_3}=\frac{1}{12}=\frac{1}{3\cdot 4}$ $n=3 \Rightarrow \sqrt{x_4}+\sqrt{x_3}=\frac{2}{15} \Rightarrow \sqrt{x_4}=\frac{1}{20}=\frac{1}{4\cdot 5}$; Demonstrăm prin inducţie matematică că $\sqrt{x_n}=\frac{1}{n\cdot(n+1)}$, $n\geq 1$	3p
	I. $\mathrm{P}(1)$: $\sqrt{x_1}=\frac{1}{1\cdot 2}(A)$ II. Presupunem că $\sqrt{x_n}=\frac{1}{n\cdot(n+1)}$, să demonstrăm că $\sqrt{x_{n+1}}=\frac{1}{(n+1)\cdot(n+2)}$ Din relaţia de recurenţă, obţinem:	1p
	$\sqrt{x_{n+1}}=\frac{2}{n\cdot(n+2)}-\frac{1}{n\cdot(n+1)}=\frac{n}{n\cdot(n+1)\cdot(n+2)}=\frac{1}{(n+1)\cdot(n+2)}$. Conform metodei inducţiei matematice $\sqrt{x_n}=\frac{1}{n\cdot(n+1)} \Rightarrow$ $x_n=\frac{1}{n^2\cdot(n+1)^2}$	1p
	$(2k+1)\cdot x_k=\frac{2k+1}{k^2\cdot(k+1)^2}=\frac{k^2+2k+1-k^2}{k^2\cdot(k+1)^2}=\frac{1}{k^2}-\frac{1}{(k+1)^2}$; $\sum_{k=1}^{n}(2k+1)\cdot x_k=\sum_{k=1}^{n}\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right)=1-\frac{1}{(n+1)^2}<1$.	2p

4.	a) "$\Rightarrow$" Fie A, B, C coliniare $\Rightarrow \overrightarrow{AB}, \overrightarrow{BC}$ sunt coliniari $\Rightarrow \overrightarrow{AB} = k \cdot \overrightarrow{BC} \Rightarrow$ $\overrightarrow{OB} - \overrightarrow{OA} = k \cdot \left(\overrightarrow{OC} - \overrightarrow{OB}\right) \Rightarrow \overrightarrow{OB} = \frac{1}{k+1} \cdot \left(\overrightarrow{OA} + k \cdot \overrightarrow{OC}\right)$. Fie $\alpha = \frac{1}{k+1} \Rightarrow 1-\alpha = \frac{k}{k+1}$. Atunci $\overrightarrow{OB} = \alpha \cdot \overrightarrow{OA} + (1-\alpha) \cdot \overrightarrow{OC}$ "$\Rightarrow$" Dacă $\overrightarrow{OB} = \alpha \cdot \overrightarrow{OA} + (1-\alpha) \cdot \overrightarrow{OC} \Rightarrow \overrightarrow{OB} - \overrightarrow{OC} = \alpha \cdot \left(\overrightarrow{OA} - \overrightarrow{OC}\right) \Rightarrow$ $\overrightarrow{CB} = \alpha \cdot \overrightarrow{CA} \Rightarrow A, B, C$ coliniare. *b*) $\overrightarrow{AA'} + \overrightarrow{AB'} + \overrightarrow{AC'} = 3 \cdot \overrightarrow{AG'}$, G' este centrul de greutate al triunghiului $A'B'C'$. La fel $\overrightarrow{BA'} + \overrightarrow{BC'} + \overrightarrow{BB'} = 3 \cdot \overrightarrow{BG'}$ şi $\overrightarrow{CA'} + \overrightarrow{CB'} + \overrightarrow{CC'} = 3 \cdot \overrightarrow{CG'}$. Adunând cele 3 egalităţi, obţinem: $3 \cdot \left(\overrightarrow{AG'} + \overrightarrow{BG'} + \overrightarrow{CG'}\right) = -3 \cdot \left(\overrightarrow{G'A} + \overrightarrow{G'B} + \overrightarrow{G'G}\right) = -9\overrightarrow{G'G}$, G este centrul de greutate al triunghiului ABC. Suma vectorilor este nulă $\Leftrightarrow G = G'$.	**2p** **1p** **1p** **1p** **1p** **1p**